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ABSTRACT

This paper is a discussion of algebraic factoring. It is suggested that factoring is frequently presented in textbooks and by teachers without specifying one of the great values of this technique; that is, finding some of the factors of any number that can be generated by a particular polynomial. Concrete examples can help students understand the value of these algebraic manipulations. As such an example, it is argued that polynomials are just an abstract way of representing an infinite set of numbers. Hence, the polynomial y cubed plus 1 cubed (where y is a positive integer) is a general way of representing 1, 2, 9, 28, 65, 126, 217, 344, 513, 730, 1001, etc. This paper is concerned only with positive integral numbers since only positive integral factors are being sought. (PK)

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Investigations in Factoring

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Investigations in Factoring

One of the major goals of algebra is that students learn to manipulate algebraic expressions. Students learn various methods of finding the roots of quadratic equations. They are taught to check first if there is an easy factorization of the polynomial before using the quadratic formula. According to Usiskin (1985), at least 30% of the first-year algebra course is spent factoring trinomials and other polynomials. Although some educators (Usiskin, 1980; Coxford, 1985) disagree with the emphasis on trinomial factoring, algebra teachers and texbooks dedicate a great amount of time to teaching and practicing this type of factoring.

Factoring is frequently presented in texbooks and by teachers without specifying one of the great values of this technique; that is, finding some of the factors of any number that can be generated by a particular polynomial. Concrete examples can help students understand the value of these algebraic manipulations.

Polynomials are just an abstract way of representing an infinite set of numbers. Hence, the polynomial $y^3 + 1^3$ (where y is a positive integer) is a general way of representing 1, 2, 9, 28, 65, 126, 217, 344, 513, 730, 1001, etc. In this article we will be concerned only with positive integral numbers since we want to find only positive integral factors.

At the beginning of a lesson in which the objective might be to teach students to factor the sum or difference of two cubes, the teacher could start the class by asking students whether 1001 is a prime or composite number. Since students have studied divisibility rules in the seventh or eighth grade, they should be able to apply the divisibility rules that they previously learned to the number in question. The most frequently taught divisibility rules are for numbers divisible by 2, 3, 4, 5, 6, 8, 9, and 10. Divisibilty rules by 7 and 11 are rarely taught, except for enrichment. Therefore, students may have a hard time deciding whether 1001 is prime or composite.

Now, 1001 can be represented as the sum of two cubes, i.e., $1001 = 10^3 + 1^3$, hence it can be represented by the polynomial $y^3 + 1$. If we can factor this polynomial we could find some of the factors of 1001. The teacher would then proceed to explain the factorization of the sum of two cubes represented as $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$. Therefore, the factorization of $y^3 + 1 = (y + 1)(y^2 - y + 1)$. When y = 10, 1001 is then factored into

$$1001 = 10^3 + 1^3 = (10 + 1)(100 - 10 + 1) = (11)(91).$$

An interesting observation is that though $(y + 1)$ and $(y^2 - y + 1)$ are called prime factors of $y^3 + 1$, the numerical values obtained need not be prime.

For the vast majority of students that do not know the divisibility rule for 11, this approach may be a much faster way of finding factors for 1001, than by a trial and error

method. But the benefit of this approach is not limited to 1001, in fact, any number that can be represented by the polynomial $y^3 + 1$ has at least two factors, one of them being the number substituted for y plus 1 and the other, $(y^2 - y + 1)$. For example, the number 217 can be represented as $6^3 + 1$, and therefore has 7 and 31 as factors. So here lies one of the beauties of mathematics, being able to factor any number that can be represented by a particular polynomial that can be factored.

Students can be encouraged to investigate the graphic representation of the two factors of $y^3 + 1$ for the different numbers it can represent. Figure 1 shows the plotting of points corresponding to pairs of factors for values of y from 0 to 10 ($y + 1$, $y^2 - y + 1$). These points suggest that the ordered pair of factors of $y^3 + 1$ lie along a parabola.

How can we find the algebraic formula for this parabola? This is an excellent exercise for algebra II and III students because it requires them to apply and practice their skill at finding the equation of a parabola. Two approaches can be used. The first one will yield the equation of the parabola but it will not help to explain why the curve will necessarily be a parabola. The second approach is much easier and the student can understand why the curve must necessarily be a parabola.

The first approach uses the general equation of a parabola, i.e., $y = ax^2 + bx + c$. Since we have three

unknowns (a, b, c), we need three points of the parabola to establish a system of equations in three unknowns. The following pairs of factors will be used: (2,1); (4, 7); and (5,13). The first coordinate of the ordered pairs represents the values that will be substituted for x, the second coordinate represents the corresponding y value. Substituting the coordinate values into the equation of the parabola gives

	(x,y)
$4a + 2b + c = 1$	----> (2,1)
$16a + 4b + c = 7$	----> (4,7)
$25a + 5b + c = 13$	----> (5,13)

Applying matrix algebra to this system of equations results in $a = 1$, $b = -3$, and $c = 3$. Hence, the equation of the suggested parabola is $y = x^2 - 3x + 3$. The inherent value of this parabola is that it will generate pairs of factors for any number represented by the polynomial $y^3 + 1$. It is interesting to note that the accompanying parabola for $y^3 - 1$ is $y = x^2 + 3x + 3$. Results of other sums or differences of cubes reflect the same pattern:

polynomial	accompanying parabola
$y^3 + 8$	$x^2 - 6x + 12$
$y^3 - 8$	$x^2 + 6x + 12$
$y^3 + 125$	$x^2 - 15x + 75$
$y^3 - 125$	$x^2 + 15x + 75$

The explanation for this mathematical occurence lies in the

second approach to finding the equation of the suggested parabola.

The second approach requires a change of variables in the factorization of $y^3 + 1$, i.e., $y + 1$ and $y^2 - y + 1$. Let $z = y + 1$, then $y = z - 1$. Substituting these values into the factorization of $y^3 + 1$ gives

$$(y + 1)(y^2 - y + 1)$$

$$z[(z - 1)^2 - (z - 1) + 1]$$

$$z(z^2 - 3z + 3)$$

This simple procedure for changing variables not only provides us with the factors of the polynomial in question but it immediately gives us the equation of the suggested parabola. Since z is the abcissa and $z^2 - 3z + 3$ is the ordinate in the ordered pair of factors, one can readily see why the curve must be a parabola.

Investigations like these involve transfer skills. Students must gather all their algebra knowledge to reach the outlined conclusion. It is a problem-solving task which requires knowledge of graphing curves, recognition of different kinds of curves, finding the equation of a curve, solving systems of equations, and performing a change of variables. Starting from the concrete to the abstract helps students experience the value of mathematics.

References

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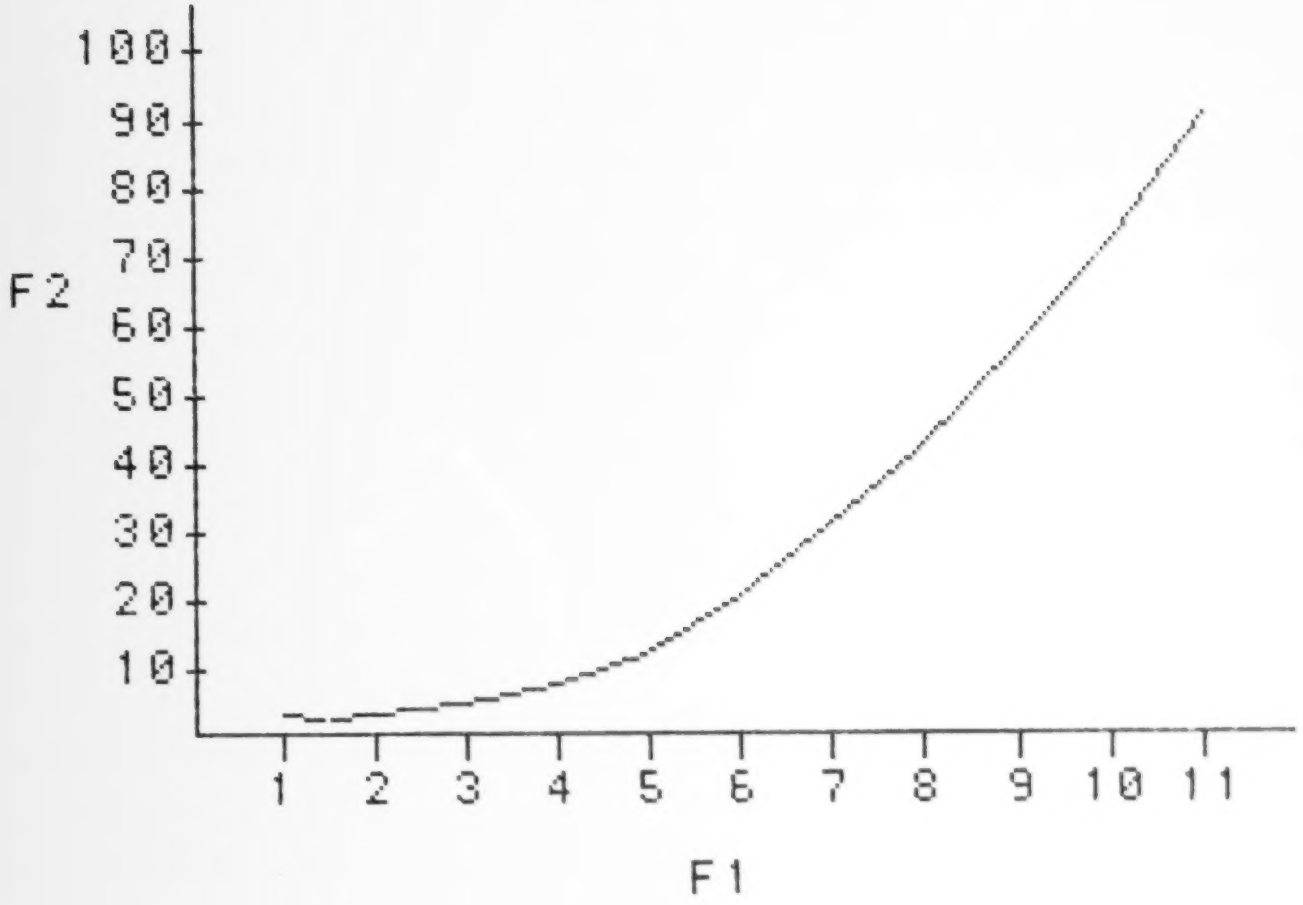
Figure 1



Vertex (3/2, 3/4)

y	$y^3 + 1$	F1: $y + 1$	F2: $y^2 - y + 1$
0	1	1	1
1	2	2	1
2	9	3	3
3	28	4	7
4	65	5	13
5	126	6	21
6	217	7	31
7	344	8	43
8	513	9	57
9	730	10	73
10	1001	11	91